The Real Zeros of a Polynomial Function

Division Algorithm for Polynomials:

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \text{ or } f(x) = q(x)g(x) + r(x).$$

$f(x)$ is the ***dividend***, $g(x)$ is the ***divisor***, $q(x)$ is the ***quotient***, and $r(x)$ is the ***remainder***.

Remainder Theorem: If a polynomial function $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.

Example: Find the remainder if $f(x) = 2x^3 - 3x^2 + 1$ is divided by

a) $x+1$

$f(-1) = 2(-1)^3 - 3(-1)^2 + 1$
$= -2 - 3 + 1 = \boxed{-4}$

b) $x-3$

$f(3) = 2(3)^3 - 3(3)^2 + 1$
$= 54 - 27 + 1 = \boxed{28}$

Factor Theorem: If f is a polynomial function, then $x-c$ is a factor of $f(x)$ if and only if $f(c) = 0$. The Factor Theorem means that:

a) If $f(c) = 0$, then $x-c$ is a factor of $f(x)$.
b) If $x-c$ is a factor of $f(x)$, then $f(c) = 0$.

Example: Use the factor theorem to determine whether the function $f(x) = 4x^4 - 15x^2 - 4$ has the factor

a) $x-2$

$f(2) = 4(2)^4 - 15(2)^2 - 4$
$= 64 - 60 - 4 = 0$
$\boxed{\text{Factor}}$

b) $x+1$

$f(-1) = 4(-1)^4 - 15(-1)^2 - 4$
$= 4 - 15 - 4 = -15$
$\boxed{\text{Not a factor}}$

Number of Real Zeros Theorem: A polynomial function cannot have more real zeros than its degree.

Rational Zeros Theorem: Let f be a polynomial function of degree 1 or higher, where each coefficient is an integer. The possible rational zeros are all numbers $\frac{p}{q}$, where p is a factor of the constant, and q is a factor of the leading coefficient of the function.

Example: List the potential rational zeros of $f(x) = 3x^4 + 4x^3 + 7x^2 + 8x + 2$.

Factors of 2: $\pm 1, \pm 2$
Factors of 3: $\pm 1, \pm 3$

potential zeros: $\frac{\pm 1}{\pm 1}, \frac{\pm 2}{\pm 1}, \frac{\pm 1}{\pm 3}, \frac{\pm 2}{\pm 3}$

$\boxed{\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}}$

Example: Find the rational zeros of $f(x) = 3x^4 + 4x^3 + 7x^2 + 8x + 2$.

Test: $f(1) = 3(1)^4 + 4(1)^3 + 7(1)^2 + 8(1) + 2 = 24$
$f(-1) = 3(-1)^4 + 4(-1)^3 + 7(-1)^2 + 8(-1) + 2 = 0$
$f(2) = 3(2)^4 + 4(2)^3 + 7(2)^2 + 8(2) + 2 = 126$
$f(-2) = 3(-2)^4 + 4(-2)^3 + 7(-2)^2 + 8(-2) + 2 = 30$
$f(\frac{1}{3}) = 3(\frac{1}{3})^4 + 4(\frac{1}{3})^3 + 7(\frac{1}{3})^2 + 8(\frac{1}{3}) + 2 = \frac{152}{27}$
$f(-\frac{1}{3}) = 3(-\frac{1}{3})^4 + 4(-\frac{1}{3})^3 + 7(-\frac{1}{3})^2 + 8(-\frac{1}{3}) + 2 = 0$
$f(\frac{2}{3}) = 3(\frac{2}{3})^4 + 4(\frac{2}{3})^3 + 7(\frac{2}{3})^2 + 8(\frac{2}{3}) + 2 = \frac{110}{9}$
$f(-\frac{2}{3}) = 3(-\frac{2}{3})^4 + 4(-\frac{2}{3})^3 + 7(-\frac{2}{3})^2 + 8(-\frac{2}{3}) + 2 = -\frac{22}{27}$

Rational zeros: $\boxed{-1, -\frac{1}{3}}$

Theorem: Every polynomial function with real coefficients can be uniquely factored over the real numbers into a product of linear factors and/or irreducible (prime) quadratic factors.

Finding the Real Zeros of a Polynomial Function:

1. Use the degree of the polynomial to determine the maximum number of real zeros.
2. Use the Rational Zeros Theorem to identify the possible rational zeros.
3. Use substitution, synthetic division, or long division to test each potential rational zero.
4. Each time a zero is found, use synthetic division to rewrite the function in factored form. Repeat step 3 on the ***depressed function*** (the function that remains after factoring out a linear factor).
5. Once the depressed function is quadratic, you can factor it or use the quadratic formula to find the remaining zeros.

★ Don't forget to use factoring techniques you already know to help you find the zeros.

Example: Find all of the real zeros of $f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$. Use the zeros to factor f over the real numbers.

Max zeros: 4

Factors of 2: $\pm 1, \pm 2$

Factors of 2: $\pm 1, \pm 2$

Potential Rational Zeros: $\pm 1, \pm 2, \pm \frac{1}{2}$

$f(1) = 0$ ✓ $\quad f(-\frac{1}{2}) = 0$

$$\begin{array}{r|rrrrr} 1 & 2 & -1 & -5 & 2 & 2 \\ & & 2 & 1 & -4 & -2 \\ \hline & 2 & 1 & -4 & -2 & 0 \end{array}$$

$f(x) = (x-1)(2x^3 + x^2 - 4x - 2)$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & 1 & -4 & -2 \\ & & -1 & 0 & 2 \\ \hline & 2 & 0 & -4 & 0 \end{array}$$

$f(x) = (x-1)(x+\frac{1}{2})(2x^2 - 4)$

$\rightarrow f(x) = 2(x-1)(x+\frac{1}{2})(x^2-2)$

$\boxed{f(x) = 2(x-1)(x+\frac{1}{2})(x+\sqrt{2})(x-\sqrt{2})}$

$\boxed{\text{zeros: } 1, -\frac{1}{2}, -\sqrt{2}, \sqrt{2}}$

Example: Find all real solutions of the equation $2x^3 - 3x^2 - 3x - 5 = 0$.

Max zeros: 3

Factors of -5: $\pm 1, \pm 5$

Factors of 2: $\pm 1, \pm 2$

Potential zeros: $\pm 1, \pm 5, \pm \frac{1}{2}, \pm \frac{5}{2}$

$f(1) = -9$

$f(-1) = -7$

$f(5) = 155$

$f(-5) = -315$

$f(\frac{1}{2}) = -7$

$f(-\frac{1}{2}) = -\frac{9}{2}$

$f(\frac{5}{2}) = 0$

$$\begin{array}{r|rrrr} \frac{5}{2} & 2 & -3 & -3 & -5 \\ & & 5 & 5 & 5 \\ \hline & 2 & 2 & 2 & 0 \end{array}$$

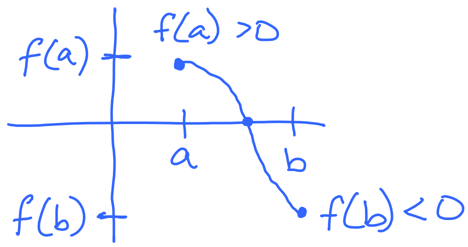
$(x - \frac{5}{2})(2x^2 + 2x + 2) = 0$

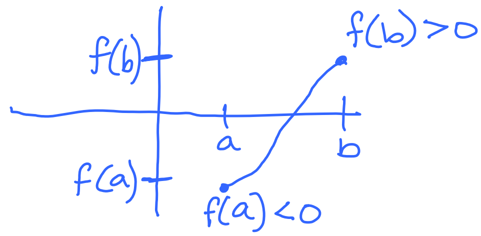
$2(x - \frac{5}{2})(x^2 + x + 1) = 0$

$b^2 - 4ac < 0 \Rightarrow$ no real solutions

$\boxed{x = \frac{5}{2}}$ is the only real zero.

Intermediate Value Theorem: For polynomial functions, if $a < b$, and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one real zero of f between a & b.





Example: Use the Intermediate Value Theorem to show that $f(x) = x^4 + 8x^3 - x^2 + 2$ has a zero in the interval $[-1, 0]$.

$$f(-1) = (-1)^4 + 8(-1)^3 - (-1)^2 + 2 = -6$$
$$f(0) = 0^4 + 8(0)^3 - 0^2 + 2 = 2$$

opposite signs, so there is a zero between -1 & 0.

The only way to get from here to here is to cross the x-axis.